

Instability of a Nielsen-Olesen Vortex Embedded in the Electroweak Theory:

II. Electroweak Vortices and Gauge Equivalence[†]

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Abstract. Vortex configurations in the electroweak gauge theory are investigated. Two gauge-inequivalent solutions of the field equations, the Z and W vortices, have previously been found. They correspond to embeddings of the abelian Nielsen-Olesen vortex solution into a U(1) subgroup of $SU(2) \times U(1)$. It is shown here that any electroweak vortex solution can be mapped into a solution of the same energy with a vanishing upper component of the Higgs field. The correspondence is a gauge equivalence for all vortex solutions except those for which the winding numbers of the upper and lower Higgs components add to zero. This class of solutions, which includes the W vortex, instead corresponds to a singular solution in the one-component gauge. The results, combined with numerical investigations, provide an argument against the existence of other vortex solutions in the gauge-Higgs sector of the Standard Model.

1. Introduction. The electroweak $SU(2) \times U(1)$ gauge theory [1] is known to admit at least two distinct vortex solutions, the Z vortex [2] and the W vortex [3]. For each of the solutions, a subset of the fields satisfies the field equations of an abelian Nielsen-Olesen vortex [4], while the other fields satisfy their equations trivially. The Z vortex, for example, is represented by an azimuthal Z field $Z_\varphi(\rho)$ and a lower component of the Higgs field $\Phi_2 = \Phi(\rho) \exp(in\varphi)$ which together satisfy the field equations of the abelian Higgs model. Here (ρ, φ) are polar coordinates of the position vector $\boldsymbol{\rho}$ perpendicular to the vortex, and n is the winding number.

For the physical value of the Weinberg angle, $\sin^2 \theta_w \approx 0.23$, the Z vortex of winding number $n = 1$ has been shown to be unstable with respect to perturbations in the charged W field [5, 6] with angular momentum $m = -1$, corresponding to the pair production of oppositely charged W bosons with angular momenta $(m, -m)$. This instability comes about because of the interaction of the Z field strength with the anomalous magnetic moment of the W boson.

Following suggestions that the instability of the Z vortex might lead to a vortex state with a condensate of W-boson pairs [6, 7] similar to the condensate formed in a strong

[†]Submitted to Mod. Phys. Lett. A as “*Electroweak Vortices and Gauge Equivalence.*”

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uniform magnetic field [8], a search for new vortex-like solutions with cylindrical symmetry was initiated. In one of these searches, undertaken by Achúcarro et al. [9], both components of the Higgs doublet were allowed to vary, leading to a gauge redundancy. Numerical solutions including W fields were found for $n > 1$ and $m = -1$, but they were able to show that the solution for each n is gauge equivalent to a Z vortex with winding number $n - 1$.

In a previous paper [10] we have investigated both the instability of the Z vortex and the existence of solutions with W fields in a different gauge, defined by the condition that the upper component of the Higgs doublet vanishes. In this one-component gauge, we showed analytically that the Z vortex with a general winding number n is (in a certain domain of the parameters β, γ) unstable under W-production in a state of angular momentum m such that $-2n < m < 0$. On the other hand, it was demonstrated that solutions with W fields can exist, in this gauge, only for m outside this range.

The purpose of the present paper is to establish an energy-preserving correspondence between vortex solutions in the two-component gauge and those in the one-component gauge. It will be shown that any two-component vortex solution with winding number n , with the exception of $m = -2n$ solutions, is gauge equivalent to a regular solution in the one-component gauge with winding number either n or $m + n$. This is a generalization of the result obtained in Ref. [9] for $m = -1$. The class of solutions with $m = -2n$, which includes the W vortex, instead corresponds to solutions in the one-component gauge with the same energy, but with SU(2) vector potentials behaving as $1/\rho$ near the origin. The solutions in the two gauges are related by a singular gauge transformation.

Because of this gauge equivalence of solutions the search for new solutions can be completely carried out in the one-component gauge. A numerical search for new solutions with W's in this gauge, for the phase indices $n = 1$ and allowed values of m ($m = 1, 0, -3$), was done by the authors with negative results. Using both variational methods and numerical methods for solution of the non-linear differential equations, only the already known Z_{NO} solutions were reproduced. This gives a rather strong indication that the gauge-Higgs sector of the Electroweak Theory admits only the already known Z and W vortex solutions.

2. Nonabelian Vortex. Let g, g' be the coupling constants for the groups SU(2) and U(1) respectively. They are related to the Weinberg angle θ_w and the electromagnetic charge e by $g \sin \theta_w = g' \cos \theta_w = e$. The physical gauge fields are related to the gauge potentials \mathbf{V}^a and \mathbf{V}' associated with the groups SU(2) and U(1) by

$$\mathbf{A} = \mathbf{V}' \cos \theta_w + \mathbf{V}^3 \sin \theta_w, \quad \mathbf{Z} = -\mathbf{V}' \sin \theta_w + \mathbf{V}^3 \cos \theta_w, \quad \text{and} \quad \mathbf{W} = (\mathbf{V}^1 - i\mathbf{V}^2)/\sqrt{2}.$$

Let us define a dimensionless vector $\mathbf{r} = \rho g \Phi_0 / (\sqrt{2} \cos \theta_w) \equiv \rho M_Z$ with polar coordinates r, φ , where Φ_0 is the Higgs vacuum expectation value.

We construct the most general time-independent electroweak vortex ansatz by letting the circle at $r = \infty$ map to an arbitrary U(1) subgroup of SU(2) \times U(1) and demanding that all fields be periodic in the azimuthal angle. The Higgs field is then given by

$$\Phi = \Phi_0 R(r) \exp \left[i \left(\frac{m}{2} \mathbf{a} \cdot \boldsymbol{\sigma} + \left(n + \frac{m}{2} \right) \varphi \right) \right] \begin{pmatrix} 0 \\ 1 \end{pmatrix} \simeq \Phi_0 \begin{pmatrix} -is_1(r)e^{i(m+n)\varphi} \\ s_2(r)e^{in\varphi} \end{pmatrix}, \quad (1)$$

where $\mathbf{a} = (\sin \alpha \cos \lambda, \sin \alpha \sin \lambda, \cos \alpha)$ is a unit vector that may depend on r , $\boldsymbol{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$ is a vector of Pauli matrices, m, n are integers, $s_1 = R \sin(\alpha/2)$, $s_2 = R \cos(\alpha/2)$ and $R(\infty) = 1$.

In what follows we shall use the expression on the right-hand side which was obtained from the general expression on the left by a gauge rotation depending only on r . Cylindrical symmetry of the energy density requires the field \mathbf{W} to be of the form

$$\mathbf{W} \cos \theta_w = \Phi_0 [u(r)\mathbf{e}_r + iv(r)\mathbf{e}_\varphi] \exp(im\varphi) . \quad (2)$$

A change in the relative phase of u and v affects the radial components A_r and Z_r . It can be shown that one may choose u, v real and $A_r = Z_r = 0$ without loss of generality. Let us then introduce a set of functions X, Y, Z defined by

$$V_\varphi^3 \cos \theta_w / \sqrt{2} = \Phi_0 Y(r), \quad V_\varphi' \sin \theta_w / \sqrt{2} = \Phi_0 X(r), \quad Z_\varphi / \sqrt{2} = \Phi_0 (Y - X) = \Phi_0 Z(r). \quad (3)$$

It is also convenient to use a set of auxiliary fields

$$y = Y - \frac{m}{2r}, \quad x = X - \frac{m}{2r} - \frac{n}{r}, \quad z = Z + \frac{n}{r} = y - x \quad (4)$$

and the parameters $\beta = (M_H/M_Z)^2$, $\gamma = (M_W/M_Z)^2 = \cos^2 \theta_w$.

The energy density in terms of these static fields and the new variables \mathbf{r} takes the form

$$\begin{aligned} \mathcal{H} = & \Phi_0^2 \left\{ (\dot{s}_1 + us_2)^2 + (\dot{s}_2 - us_1)^2 + ((y+x)s_1 - vs_2)^2 + ((y-x)s_2 + vs_1)^2 \right. \\ & \left. + \frac{1}{4}\beta(s_1^2 + s_2^2 - 1)^2 + \frac{1}{\gamma}(\dot{v} + \frac{v}{r} + 2yu)^2 + \frac{1}{\gamma}(\dot{y} + \frac{y}{r} - 2uv)^2 + \frac{1}{1-\gamma}(\dot{x} + \frac{x}{r})^2 \right\}, \quad (5) \end{aligned}$$

where a *dot* indicates differentiation with respect to r . In addition to gauge invariances the action in this model is invariant under charge conjugation. This implies the invariance of the energy density under the following substitutions:

$$y \rightarrow -y, \quad x \rightarrow -x, \quad u \rightarrow -u, \quad s_1 \rightarrow -s_1 \quad \text{and} \quad (n \rightarrow -n, \quad m \rightarrow -m).$$

Consequently it is sufficient to consider only positive values of n .

3. Correspondence of Gauges. It is easy to verify that the Euler-Lagrange equations for the variational principle $\delta \int \mathcal{H} d^2\mathbf{r} = 0$ are not all independent. The equation for the field s_1 is implied by the other equations. In the case $s_1 = 0$ it becomes the integrability condition of the other equations. This means that the gauge, as expected, has not been completely fixed in our ansatz. The form of the ansatz (and the energy) is in fact invariant under two gauge transformations U_0 or U_π , where

$$U_0(\xi) = \exp(\frac{i}{2}\sigma_3 m\varphi) \exp(\frac{i}{2}\sigma_1 \xi) \exp(-\frac{i}{2}\sigma_3 m\varphi); \quad U_\pi(\xi) \equiv i\sigma_1 U_0(\xi - \pi). \quad (6)$$

For these transformations to be nonsingular the gauge parameter $\xi(r)$ must satisfy the boundary conditions $\xi(0) = 0$ or $\xi(0) = \pi$ respectively. Under the transformation U_π the phase indices m and n change to $\bar{n} = n + m$ and $\bar{m} = -m$ while for U_0 , $\bar{n} = n$ and $\bar{m} = m$. In either transformation, $\alpha \rightarrow \bar{\alpha} = \alpha - \xi$, $Y \rightarrow \bar{Y}$, $y \rightarrow \bar{y}$ and $v \rightarrow \bar{v}$, where

$$\bar{y} \equiv \bar{Y} - \frac{\bar{m}}{2r} = y \cos \xi + v \sin \xi; \quad \bar{v} = -y \sin \xi + v \cos \xi. \quad (7)$$

It is then obvious that, whenever allowing a two-component Higgs field, the gauge freedom can be used to set $\bar{v}(r) = 0$. Vice versa, by applying $U_0(\omega)^{-1}$ or $U_\pi(\omega)^{-1}$ to a configuration with $\bar{v} = 0$, y and v may always be parametrized by $y = G \cos \omega$, $v = G \sin \omega$, where

$\omega(0) = 0$ or π respectively. In this parametrization, a gauge transformation with parameter ξ corresponds to a shift $\omega \rightarrow \bar{\omega} = \omega - \xi$.

Before proceeding we shall identify the embedded Nielsen-Olesen vortex solution corresponding to the Z vortex. This Z_{NO} solution has only the lower component of the Higgs field $s_2 = f_{NO}(r) \exp(in\varphi)$ and fields $Z = -v_{NO}(r)/r$, $X = (\gamma - 1)Z$, $Y = \gamma Z$, and corresponds to $\alpha \equiv \omega \equiv 0$. Here the functions f_{NO} and v_{NO} are those defined in Ref. [5].

One finds that in the new variables R, α, G, ω the action density (5) can be written as:

$$\begin{aligned} \mathcal{H} = & \Phi_0^2 \left\{ \dot{R}^2 + R^2 \frac{\dot{\alpha}^2}{4} + \frac{1}{\gamma} \left(\dot{G} + \frac{G}{r} \right)^2 + G^2 \dot{\omega}^2 \right\} + \frac{\beta}{4} (R^2 - 1)^2 + \frac{1}{1 - \gamma} \left(\dot{x} + \frac{x}{r} \right)^2 \\ & + (x^2 + G^2) R^2 - 2R^2 G x \cos(\alpha - \omega) + \left(R^2 + \frac{4G^2}{\gamma} \right) u^2 + u \left(R^2 \dot{\alpha} + \frac{4G^2}{\gamma} \dot{\omega} \right) \} \quad (8) \end{aligned}$$

The Euler-Lagrange equation for u gives

$$\left(R^2 + \frac{4G^2}{\gamma} \right) u + R^2 \frac{\dot{\alpha}}{2} + \frac{2G^2}{\gamma} \dot{\omega} = 0 \quad (9)$$

Replacing u as given by this equation into the action one obtains, after some algebraic manipulation, the following constrained action density:

$$\begin{aligned} \mathcal{H}^{(c)} = & \Phi_0^2 \left\{ \dot{R}^2 + \frac{1}{\gamma} \left(\dot{G} + \frac{G}{r} \right)^2 + \frac{1}{1 - \gamma} \left(\dot{x} + \frac{x}{r} \right)^2 + \frac{\beta}{4} (R^2 - 1)^2 + (x^2 + G^2) R^2 \right. \\ & \left. - 2R^2 G x \cos(\alpha - \omega) + \frac{R^2 G^2}{\gamma R^2 + 4G^2} (\dot{\alpha} - \dot{\omega})^2 \right\} \quad (10) \end{aligned}$$

This expression depends on α and ω only in the combination $(\alpha - \omega)$ which is again the evidence of the lack of gauge fixing.

By setting $\alpha \equiv 0$ or $\omega \equiv 0$ one obtains two distinct gauges, one-component and two-component Higgs gauges, C1[ω] and C2[α] respectively. The argument in a square bracket indicates which field remains dynamical in each gauge. Now one can see from Eq. (10) that the Euler-Lagrange equations for the variational principle $\delta \int \mathcal{H}^{(c)} d^2 \mathbf{r}$ are the same in the two gauges C1[ω] and C2[α] provided that one identifies the variable ω in the first gauge with $\pm \alpha$ in the second gauge. This will allow us to establish a one-to-one, energy preserving correspondence between solutions of the field equations in the two gauges.

In the gauge C1[ω], if $m \neq 0$, one has to impose the boundary condition $\omega(0) = (0 \text{ or } \pi)$ in order to exclude a pole of v at $r = 0$. Then the boundary conditions for $\alpha(r)$ in the C2[α] gauge leading to a solution with non-singular physical fields should be $\alpha(0) = 0$ or $\alpha(0) = \pi$. In fact, the Euler-Lagrange equation for α in this gauge is:

$$\frac{d}{dr} \left(\frac{r R^2 G^2}{\gamma R^2 + 4G^2} \frac{d}{dr} \alpha \right) - r R^2 G x \sin \alpha = 0 \quad (11)$$

This equation admits the trivial solutions $\alpha(r) \equiv 0$ and $\alpha(r) \equiv \pi$ which correspond to the Z_{NO} solutions with only the lower component Higgs and index n or the upper component Higgs and index $l = n + m$. In the second case the solution in the first gauge will be obtained by applying the gauge transformation $U_\pi(\xi)$ with $\xi = \pi$. Let us next investigate the possibility of solutions other than those. If they exist, they would correspond to states with charged W fields.

If $m \neq (0 \text{ or } -2n)$ then near $r = 0$, $y = G \sim -m/2r$ and $x \sim -(2n + m)/2r$. Setting $R = R_0 r^p$ and $\alpha = \alpha_0 + \alpha_1 r^q$, ($q > 0$), Eq. (11) reduces to

$$r^{(2p+q-1)} q(2p+q) \alpha_1 - r^{2p-1} m(2n+m) \sin(\alpha_0 + \alpha_1 r^q) = 0 \quad (12)$$

which, since $q > 0$, implies that $\alpha(0) = \alpha_0 = (0 \text{ or } \pi)$.

From Eq. (12) we obtain necessary conditions for the existence of solutions in the C2[α] gauge.

In the case $\alpha(0) = 0$, one finds $p = n$ and either $q = m$ or $q = -(2n + m)$. Since q must be positive it follows that m must be *outside* the interval $-2n < m < 0$. This agrees with the condition for existence of solutions in the C1[ω] gauge found previously [10]. Indeed, solutions with $\alpha(0) = 0$ would be related to solutions in the C1[ω] gauge with $\omega(0) = 0$ and the same n, m by the gauge transformation $U_0(\xi)$ where $\xi(r) = \alpha = -\omega$.

In the case $\alpha(0) = \pi$ we obtain $p = |n + m|$ and $q = -|m + n| \pm n > 0$. Since n is positive, it follows that m must now satisfy $-2n < m < 0$. For any allowed m the corresponding solution in the gauge C1[ω] is obtained by the gauge transformation $U_\pi(\xi)$ where $\xi(r) = \alpha = -\omega$ (modulo 2π). It will have $\omega(0) = \pi$ and phase indices $\bar{n} = n + m$ and $\bar{m} = -m$. If one uses charge conjugation invariance to fix the sign of \bar{n} as positive, ($\bar{n} = |n + m|, \bar{m} = \pm m$) then the condition $-2n < m < 0$ in gauge C2[α] is equivalent to the condition \bar{m} *outside* the interval $-2\bar{n} < \bar{m} < 0$ in the gauge C1[ω].

If $m = 0$ there is no restriction on $\alpha(0)$ (or on $\omega(0)$ in the C1[ω] gauge) and the two gauges are equivalent, since then $U_0(\xi) \equiv U_\pi(\xi)$ is nonsingular for any value of $\xi(0)$.

If $m = -2n$, then in Eq. (12) one has to take into account the behavior $x \simeq x_1 r$ of the function $x(r)$ near $r = 0$. One finds $q = 2$ and again no restriction on $\alpha(0)$. One can show that in the one-component gauge C1[ω], if a solution with charged W fields exists, $v(r)$ must have a simple pole and $u(r)$ vanishes at $r = 0$. Since the gauge C2[α] was fixed by the condition $v(r) = 0$ under the assumption that $v(r)$ was regular in any equivalent gauge it follows that a corresponding solution in the second gauge can only be related to that in the first by a singular gauge transformation.

If $\gamma = 1$, then one must have $x(r) = X(r) = 0$ and the gauge group reduces to SU(2). A solution with $x(r) = 0$, corresponding to the embedding of the Nielsen-Olesen vortex in an SU(2) group, is then expected to exist also for $\gamma < 1$ in the SU(2) \times U(1) gauge theory. An inspection of the energy density (5) when $x(r) = 0$ shows that it is invariant under a rotation of the two-component vector \mathbf{s} defined by $\mathbf{s} = (s_1, s_2)$. In a two-component gauge one should be able to use this freedom of rotation to enforce the equation $x(r) = 0$. After obtaining the equation for x from the expression (5) for the energy density, one finds that for $\gamma < 1$, a solution $x(r) = 0$ implies the condition $2y(s_1^2 - s_2^2) - 4vs_1s_2 = 0$. A non-singular solution will require $s_1(r) = \pm s_2(r)$, $v(r) = 0$, that is, it could exist in the C2[α] gauge. Such a solution, the W vortex, was indeed found in Ref. [3]. In the C2[α] gauge and in terms of the fields R, G, x, α, ω it is given by:

$$\omega \equiv 0, \quad \alpha \equiv \pi/2, \quad x = 0, \quad R = f_{NO, \beta \rightarrow \beta/\gamma}(\sqrt{\gamma}r), \quad y = G = -v_{NO, \beta \rightarrow \beta/\gamma}(\sqrt{\gamma}r)/r + n/r.$$

We remark that the W vortex, in this gauge, contains no charged W fields. The corresponding solution with W's in the C1[ω] gauge is obtained by applying to this solution the singular gauge transformation $U_0(\frac{\pi}{2})$.

The invariance of the energy density under a rotation of the vector \mathbf{s} implies that for $\gamma < 1$ and for any β the W vortex is a saddle point, since around this extremum there

are directions in function space along which the energy does not change. This result is in agreement with the conclusion of Klinkhamer and Olesen [11].

One concludes from this analysis that except when $n + l = 2n + m = 0$, every regular solution of the Euler-Lagrange equations in a two-component gauge, with given values of the indices (n, m) is gauge equivalent to a solution in the one-component gauge with either indices $(\bar{n} = n, \bar{m} = m)$ or $(\bar{n} = n + m, \bar{m} = -m)$. This generalizes a result of Achúcarro et al. [9] who found numerically a two-component solution with charged W fields for the special case $m = -1$ and showed that it was gauge equivalent to the Z_{NO} solution with index $n - 1$.

In the exceptional case $2n + m = 0$, a solution with charged W's in the gauge $C1[\omega]$ is always singular and can only be related to a regular solution in the $C2[\alpha]$ gauge by a singular gauge transformation.

Regular solutions with charged W's in the one-component gauge must have m outside the interval $-2n \leq m < 0$. We have made a rather thorough numerical search for solutions with W's in this gauge for $n = 1$ and $m = 1, 0, -3$ with negative results. This seems to indicate that the Z vortex and the W vortex already found are the only possible vortex solutions in the gauge-Higgs sector of the $SU(2) \times U(1)$ Electroweak Theory.

One of us, O.T., would like to thank Poul Olesen for helpful discussions.

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